Final exam of Topology II
Due Date: March 9, 2018

1. Let X be the space of 2-dimensional subspaces of $\mathbb{R}^5$. (This is called a Grassmanian, and is sometimes denoted $\mathrm{Gr}(5,2)$). Find appropriate charts that give X the structure of a smooth manifold.

2. Show that every non-orientable manifold has a 2-sheeted orientable covering space.

3. Suppose that M be a compact, orientable n-manifold without boundary and let θ be a $(n-1)$-form. Prove that $d\theta$ must vanish at some point.

4. Let y be a regular value of a differentiable map $f : X \to Y$ where X is compact and X and Y are differentiable manifolds of the same dimension. Show that there exists an open neighborhood U of y such that $f^{-1}(U)$ is a disjoint union of finitely many open sets V_i, $i = 1, \cdots, n$, and, for each i, $f|_{V_i}$ is a diffeomorphism with U.

5. Let ω be an r-form on a differentiable manifold M. Assume that $\int_\Sigma \omega = 0$ for every submanifold Σ of M diffeomorphic to S^r. Show that ω is closed.

6. Prove or disprove. Let M be a connected manifold and $F \subset M$ a finite subset. Then there exists a smooth vector field X such that X restricts to nonzero at each point in F.

7. Show that if the map $f : S^n \to \mathbb{R}$ is differentiable, then there exist two different points $p, q \in S^n$, so that df_p and df_q are both 0.

8. (a) Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every continuous map $X \to S^1$ is homotopic to a constant map.

   (b) Let $p : E \to B$ be a covering map, with E path connected. Show that if B is simply-connected, then p is a homeomorphism.